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ELASTIC FIELDS OF INTERACTING INHOMOGENEITIES

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Abstract—A rather general technique—called the "method of pseudotractions"—is presented for the calculation of the stress and strain fields in a linearly elastic homogeneous solid which contains any number of defects of arbitrary shape. The method is introduced and illustrated in terms of the problems of elastic solids containing two or several circular holes and solids containing two or several cracks, including the cases of rows of holes or cracks. It is shown that the solution of these and similar problems can be obtained to any desired degree of accuracy. Furthermore, if only estimates are needed, then the method is capable of yielding closed-form analytic expressions for many interesting cases, e.g. the stress intensity factors at the crack tips.

1. INTRODUCTION

The nonlinear and history-dependent responses of many solids most often stem from the presence of microdefects such as inhomogeneities, voids, and cracks, and from the evolutionary changes of these defects in the course of a given load history. To estimate the influence of microdefects on the overall material response, many techniques have been developed; see Mura[1] for an account and a comprehensive list of references. Most commonly used approaches are based on the stress and strain fields in an infinitely extended, linearly elastic, homogeneous solid containing an isolated microdefect. This does not account for the interaction between neighboring defects. Another approach has been to assume a periodic distribution of microdefects, which may tend to overestimate the interaction effect. An account of this and references to recent works are given by Nemat-Nasser et al.[2] and Iwakuma and Nemat-Nasser[3].

The purpose of the present study is to give a rather general technique for the calculation of the stress and strain fields in a linearly elastic, homogeneous solid which contains any number of defects of arbitrary shape, provided that we have the solutions of a set of subsidiary problems, each consisting of only one defect in an infinitely extended solid. The method presented here is limited to two-dimensional problems, although, in principle, one may try to develop a similar approach for three-dimensional cases. For two-dimensional problems, however, Muskhelishvili's[4] complex stress potentials provide an effective tool in the calculations, whereas different mathematical techniques must be employed for three-dimensional cases.

To be explicit, we shall introduce our method—which we have called the "method of pseudotractions"—in terms of several specific examples in Sections 3 and 4; in Section 2 we list some of the basic equations. In Section 3, we consider a solid containing two circular holes, then generalize the results to the case when there are several holes, and finally, we examine the case of a row of periodically distributed holes. In Section 4, we consider a solid with two interacting cracks and then generalize our results to several interacting cracks. The solution of these and similar problems can be obtained to any desired degree of accuracy. Furthermore, if only an estimate is needed, then the method is capable of yielding closed-form analytical expressions for many interesting cases, e.g. the stress intensity factors at the tips of several arbitrarily shaped interacting cracks. Recently, this approach has been effectively used by Horii and Nemat-Nasser[5] to study the micromechanics of the failure mechanisms in brittle solids under compression, containing microdefects.

2. BASIC EQUATIONS

The boundary-value problems which will be dealt with in this work are most effectively formulated in terms of the complex stress potentials Φ and Ψ of Muskhelishvili[4]. With respect to a rectangular Cartesian coordinate system x, y, the normal stresses σ_x and σ_y, the shear stress $\tau_{xy} = \tau_{yx}$, and the displacement components u_x and u_y, are given by

$$\begin{aligned} \sigma_x + \sigma_y &= 2(\Phi' + \overline{\Phi'}), \\ \sigma_y - \sigma_x + 2i\tau_{xy} &= 2(\bar{z}\Phi'' + \Psi'), \\ 2G(u_x + iu_y) &= \kappa\Phi - z\overline{\Phi'} - \overline{\Psi}, \\ z = x + iy, &\qquad i = \sqrt{-1}, \end{aligned} \tag{2.1}$$

where G is the shear modulus, and $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress, ν being Poisson's ratio; the over-bar denotes the complex conjugate, and prime stands for differentiation with respect to the argument. With respect to the polar cooordinates θ, r, the hoop stress σ_θ and the shear stress $\tau_{r\theta}$ become

$$\sigma_\theta + i\tau_{r\theta} = \Phi' + \overline{\Phi'} + e^{2i\theta}(\bar{z}\Phi'' + \Psi'). \tag{2.2}$$

Suppose C is a straight crack on the x-axis, with end points c_1 and c_2. Let on C the following boundary condition be prescribed:

$$\sigma_y - i\tau_{xy} = p(x) \text{ on } C. \tag{2.3}$$

Then, the corresponding stress potentials are given by

$$\begin{aligned} \Phi'(z) &= \frac{1}{2\pi i X(z)} \int_{c_1}^{c_2} \frac{X(t)p(t)}{t - z}\,dt + \frac{P(z)}{X(z)}, \\ \Psi'(z) &= \overline{\Phi'(\bar{z})} - \Phi'(z) - z\Phi''(z), \end{aligned} \tag{2.4}$$

where integration is on C, $P(z)$ is a polynomial, and, for solutions with unbounded stresses at the end points, we have

$$X(z) = [(z - c_1)(z - c_2)]^{1/2}. \tag{2.5}$$

$P(z)$ in eqn (2.4) characterizes the order of the pole of $\Phi'(z)$ at infinity. This polynomial is fixed in such a manner that the prescribed conditions at infinity are satisfied and the displacement field is rendered single-valued.

When C is a circular cavity of radius a, subjected to self-equilibrating stresses, $\sigma_r + i\tau_{r\theta} = p(t)$, $\Phi'(z)$ and $\Psi'(z)$ are given by

$$\begin{aligned} \Phi'(z) &= -\frac{1}{2\pi i} \int_C \frac{p(t)}{t - z}\,dt + A_0 + A_1/z^2, \\ \Psi'(z) &= (a/z)^2[\Phi'(z) + \overline{\Phi'(a^2/\bar{z})} - z\Phi''(z)], \end{aligned} \tag{2.6}$$

where integration is on C in a counterclockwise manner, and A_0 and A_1 are complex constants, fixed in such a manner that the prescribed conditions at infinity are satisfied.

3. AN INFINITELY EXTENDED SOLID WITH CIRCULAR HOLES

3.1 *A solid with two holes*

As a first example, consider an infinitely extended solid containing two circular holes, as shown in Fig. 1, under farfield uniform stresses. Let x^1, y^1 and x^2, y^2 be two parallel rectangular Cartesian coordinate systems with origins 0^1 and 0^2 at the centers of holes 1 and 2, respectively. Denote by r^1, θ^1 and r^2, θ^2 the associated polar coordinate systems. Let the distance between 0^1 and 0^2 be d^{21}, and the angle measured from the

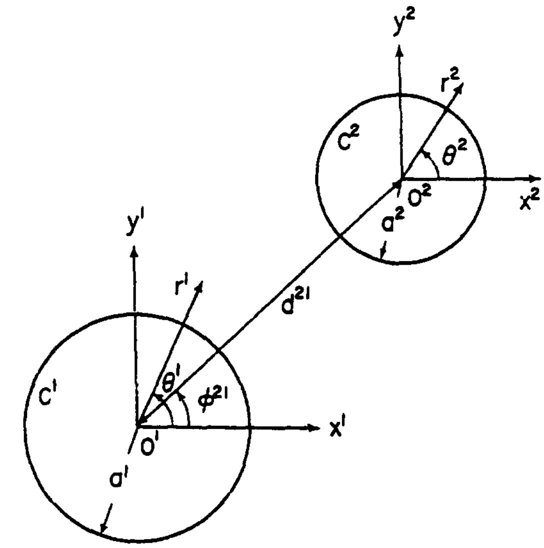


Fig. 1. An infinitely extended plate with two circular holes.

x^1-direction to the 0^10^2-direction be denoted by ϕ^{21}. The components in the x^jy^j- and $r^j\theta^j$-coordinates are indicated with the superscript j, $j = 1, 2$. The quantities followed by superscript ∞ denote values at infinity. The surfaces of the holes are stress-free. We shall refer to this boundary-value problem as the "original problem."

The solution of this problem is obtained by the superposition of the solutions of a homogeneous problem and two sub-problems, denoted by 1 and 2; see Fig. 2. In the homogeneous problem, an infinitely extended body without any holes is subjected to the applied stresses at infinity. Sub-problems 1 and 2 each consist of an infinitely extended body with only one hole with zero stresses at infinity. In sub-problem j, the boundary conditions along the surface, C^j, of hole j are given by

$$\sigma_r^j + \sigma_r^{\infty j} + \sigma_r^{Pj} = 0, \; \tau_{r\theta}^j + \tau_{r\theta}^{\infty j} + \tau_{r\theta}^{Pj} = 0, \text{ on } C^j, \qquad j = 1, 2. \tag{3.1}$$

The quantities σ_r^{P1}, $\tau_{r\theta}^{P1}$, σ_r^{P2}, and $\tau_{r\theta}^{P2}$ will be called "pseudotractions." They are the unknown functions which must be determined in such a manner that all boundary

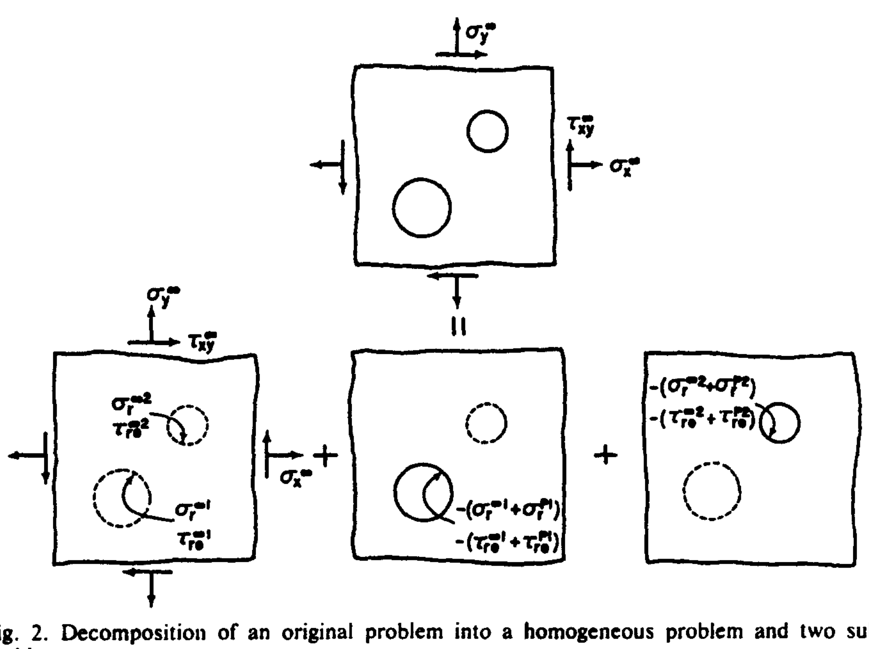


Fig. 2. Decomposition of an original problem into a homogeneous problem and two sub-problems.

conditions of the "original problem" are satisfied. From the equilibrium conditions, the pseudotractions must satisfy

$$\int_0^{2\pi} (\sigma_r^{Pj} + i\tau_{r\theta}^{Pj})e^{i\theta^j}\, d\theta^j = 0, \qquad j = 1, 2. \tag{3.2}$$

The Muskhelishvili[4] stress functions for the sub-problem j are given by

$$\begin{aligned}\Phi^{j\prime}(z^j) &= \frac{1}{2\pi i}\int_{C^j} \frac{\sigma_r^{\infty j} + \sigma_r^{Pj} + i(\tau_{r\theta}^{\infty j} + \tau_{r\theta}^{Pj})}{t - z^j}\, dt,\\ \Psi^{j\prime}(z^j) &= (a^j/z^j)^2[\Phi^{j\prime}(z^j) + \overline{\Phi^{j\prime}(a^{j2}/\bar{z}^j)} - z^j\Phi^{j\prime\prime}(z^j)], \qquad j = 1, 2,\end{aligned} \tag{3.3}$$

where $z^j = x^j + iy^j$. The requirement that the sum of the solutions of the sub-problems must be equivalent to that of the original problem leads to

$$\begin{aligned}\sigma_r^{P2} + i\tau_{r\theta}^{P2} &= \Phi^{1\prime}(z^1) + \overline{\Phi^{1\prime}(z^1)} - e^{-2i\theta^2}[z^1\overline{\Phi^{1\prime\prime}(z^1)} + \overline{\Psi^{1\prime}(z^1)}],\\ \sigma_r^{P1} + i\tau_{r\theta}^{P1} &= \Phi^{2\prime}(z^2) + \overline{\Phi^{2\prime}(z^2)} - e^{-2i\theta^1}[z^2\overline{\Phi^{2\prime\prime}(z^2)} + \overline{\Psi^{2\prime}(z^2)}],\end{aligned} \tag{3.4}$$

with

$$\begin{aligned}&z^1 = d^{21}e^{i\phi^{21}} + a^2e^{i\theta^2},\ z^2 = d^{12}e^{i\phi^{12}} + a^1e^{i\theta^1},\\ &d^{12} = d^{21}, \text{ and } \phi^{12} = \pi + \phi^{21}.\end{aligned} \tag{3.5}$$

The right-hand sides of eqns (3.4) represent the tractions acting on C^2 and C^1 in sub-problems 1 and 2, respectively. It is seen from eqns (3.1) that eqns (3.4) ensure the stress-free condition on the surfaces of the holes when the homogeneous problem and the sub-problems are superimposed.

Equations (3.2), (3.3), and (3.4) form a system of integral equations for the pseudotractions. In general, it is not easy to solve this system of integral equations explicitly. However, this system can be reduced to a system of algebraic equations in the following manner.

We expand the tractions along C^j into a Fourier series as

$$\begin{aligned}\sigma_r^{Pj} + i\tau_{r\theta}^{Pj} &= \sum_{n=-\infty}^{\infty} (P_n^j + iQ_n^j)e^{in\theta^j},\\ \sigma_r^{\infty j} + i\tau_{r\theta}^{\infty j} &= P_0^\infty + (P_{-2}^\infty + iQ_{-2}^\infty)e^{-2i\theta^j}, \qquad j = 1, 2,\end{aligned} \tag{3.6}$$

where

$$P_0^\infty = (\sigma_x^\infty + \sigma_y^\infty)/2,\ P_{-2}^\infty = (\sigma_x^\infty - \sigma_y^\infty)/2, \text{ and } Q_{-2}^\infty = \tau_{xy}^\infty. \tag{3.7}$$

It follows from eqns (3.2) that

$$P_{-1}^j = Q_{-1}^j = 0, \qquad j = 1, 2. \tag{3.8}$$

Substituting eqns (3.6) into eqns (3.3), we obtain

$$\begin{aligned}\Phi^{j\prime}(z^j) &= -\sum_{n=-\infty}^{-2} (P_n^j + iQ_n^j)(z^j/a^j)^n - (P_{-2}^\infty + iQ_{-2}^\infty)(z^j/a^j)^{-2},\\ \Psi^{j\prime}(z^j) &= -\sum_{n=-\infty}^{-2} (1 - n)(P_n^j + iQ_n^j)(z^j/a^j)^{n-2} + \sum_{n=0}^{\infty} (P_n^j - iQ_n^j)(z^j/a^j)^{-n-2}\\ &\quad - 3(P_{-2}^\infty + iQ_{-2}^\infty)(z^j/a^j)^{-4} + P_0^\infty(z^j/a^j)^{-2}, \qquad j = 1, 2.\end{aligned} \tag{3.9}$$

We substitute (3.9) into (3.6) to obtain a system of algebraic equations for the Fourier coefficients, P^j_n and Q^j_n,

$$
\begin{aligned}
P^j_n &= \sum_{m=-\infty}^{\infty} [A^{jk}_{nm}P^k_m + B^{jk}_{nm}Q^k_m] + A^{jk}_{n0}P^\infty_0 + A^{jk}_{n-2}P^\infty_{-2} + B^{jk}_{n-2}Q^\infty_{-2}, \\
Q^j_n &= \sum_{m=-\infty}^{\infty} [C^{jk}_{nm}P^k_m + D^{jk}_{nm}Q^k_m] + C^{jk}_{n0}P^\infty_0 + C^{jk}_{n-2}P^\infty_{-2} + D^{jk}_{n-2}Q^\infty_{-2},
\end{aligned}
\tag{3.10}
$$

where

$$
A^{jk}_{nm} = \begin{cases}
I^{jk}_{n\ -m}, & \text{for } n \geq 1,\ m \leq -2, \\
2I^{jk}_{0\ -m}, & \text{for } n = 0,\ m \leq -2, \\
(m-1)I^{jk}_{-n-2\ 2-m} + (n+1)I^{jk}_{-n\ -m} & \\
\quad \times \left[1 - \dfrac{n}{n+m+1}(a^j/d^{jk})^2\right], & \text{for } n \leq -2,\ m \leq -2, \\
I^{jk}_{n-2\ m+2}, & \text{for } n \leq -2,\ m \geq 0, \\
0, & \text{otherwise},
\end{cases}
$$

$$
B^{jk}_{nm} = \begin{cases}
II^{jk}_{n\ -m}, & \text{for } n \geq 1,\ m \leq -2, \\
2II^{jk}_{0\ -m}, & \text{for } n = 0,\ m \leq -2, \\
(m-1)II^{jk}_{-n-2\ 2-m} + (n+1)II^{jk}_{-n\ -m} & \\
\quad \times \left[1 - \dfrac{n}{n+m+1}(a^j/d^{jk})^2\right], & \text{for } n \leq -2,\ m \leq -2, \\
-II^{jk}_{-n-2\ m+2}, & \text{for } n \leq -2,\ m \geq 0, \\
0, & \text{otherwise},
\end{cases}
\tag{3.11}
$$

$$
C^{jk}_{nm} = \begin{cases}
0, & \text{for } n = 0,\quad m \leq -2, \\
B^{jk}_{nm}, & \text{for } n \leq -2,\ m \leq -2, \\
-B^{jk}_{nm}, & \text{otherwise},
\end{cases}
$$

$$
D^{jk}_{nm} = \begin{cases}
0, & \text{for } n = 0,\quad m \leq -2, \\
-A^{jk}_{nm}, & \text{for } n \leq -2,\ m \leq -2, \\
A^{jk}_{nm}, & \text{otherwise},
\end{cases}
$$

with

$$
\begin{Bmatrix} I^{jk}_{nm} \\ II^{jk}_{nm} \end{Bmatrix} = \frac{(-1)^{n+1}(m+n-1)!}{n!(m-1)!}(a^k/d^{jk})^n(a^j/d^{jk})^m \begin{Bmatrix} \cos(m+n)\phi^{jk} \\ \sin(m+n)\phi^{jk} \end{Bmatrix}. \tag{3.12}
$$

As is seen from eqns (3.10–3.12), the coefficients P^j_n and Q^j_n are of the order of $(a/d)^{|n+1|+1}$ where $a/d = 0(a^1/d^{21}) = 0(a^2/d^{12})$. Neglecting high order Fourier coefficients, P^j_n, Q^j_n, $n < -N-1$, and $n > N-1$, eqns (3.9) become $8N-2$ equations for $8N-2$ unknowns, $P^j_{-N-1}, \ldots, P^j_{-2}, P^j_0, \ldots, P^j_{N-1}, Q^j_{-N-1}, \ldots, Q^j_{-2}, Q^j_1, \ldots, Q^j_{N-1}$, $j = 1, 2$. Then eqns (3.9) are easily solved, and the solution of the original problem is obtained by superposition. The stress functions for sub-problems 1 and 2 are given by (3.9). For example, the hoop stress σ^j_θ along the surface of hole C^j is given by

$$
\begin{aligned}
\sigma^j_\theta = {} & \sigma^\infty_x + \sigma^\infty_y - 2[(\sigma^\infty_x - \sigma^\infty_y)\cos 2\theta^j + 2\tau^\infty_{xy}\sin 2\theta^j] \\
& + 2P^\infty_0 + 4(P^j_1\cos\theta^j - Q^j_1\sin\theta^j) \\
& + 4\sum_{n=2}^{\infty}[(P^j_n - P^j_{-n})\cos n\theta^j - (Q^j_n - Q^j_{-n})\sin n\theta^j].
\end{aligned}
\tag{3.13}
$$

Typical examples are shown in Table 1 for $\phi^{21} = 0$. These results coincide with those by Haddon[6]. Within the number of significant figures shown in Table 1, the accuracy of the solution does not change for N greater than for those indicated.

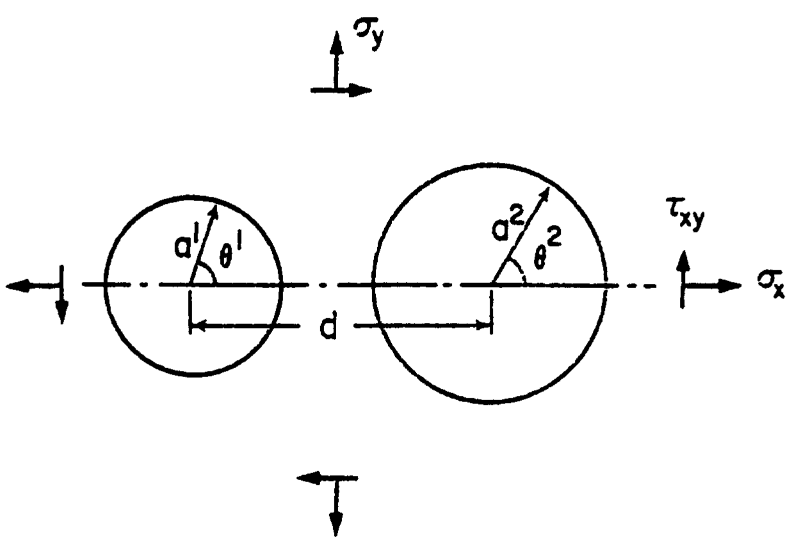
3.2 *A solid with several holes*

The problem of an infinitely extended solid containing M holes is solved in a similar manner. The original problem is decomposed into a homogeneous problem and M sub-problems where the typical sub-problem j consists of an infinitely extended solid with

Table 1. The maximum and minimum stresses in an infinite plate with two circular holes.

(i) $a^2/a^1 = 1$; $s = (d - a^1 - a^2)/a^1$

	$\sigma_x = 1$			$\sigma_y = 1$			$\sigma_x = \sigma_y = \sigma_{xy} = 0.5$		
s	σ_θ^1	θ^1	(N)	σ_θ^1	θ^1	(N)	σ_θ^1	θ^1	(N)
.4	2.6188	95.41	(22)	4.4227	0	(20)	4.4936	342.48	(22)
	−.9048	180		−.9244	89.75		−1.5220	25.18	
1	2.6500	94.60	(13)	3.2641	0	(13)	3.5867	332.01	(13)
	−.8962	180		−.8857	91.09		−1.4602	36.03	
4	2.8272	91.35	(7)	2.9922	0	(7)	3.1140	317.36	(7)
	−.9397	180		−.9202	90.63		−1.1370	44.53	
10	2.9478	90.20	(4)	2.9981	0	(4)	3.0287	315.38	(4)
	−.9790	180		−.9744	90.11		−1.0322	45.08	



(ii) $a^2/a^1 = 5$

	$\sigma_x = 1$								
s	σ_θ^1	θ^1	σ_θ^1	θ^1	σ_θ^2	θ^2	σ_θ^2	θ^2	(N)
1	.8194	106.24	−.8605	0	2.9998	90.02	−1.0041	0	(30)
4	1.5433	99.65	−.5272	180	2.9899	89.80	−.9965	0	(17)
10	2.3026	92.49	−.7238	106.24	2.9859	89.81	−.9954	0	(9)

	$\sigma_y = 1$								
s	σ_θ^1	θ^1	σ_θ^1	θ^1	σ_θ^2	θ^2	σ_θ^2	θ^2	(N)
1	6.1180	0	−.6477	84.27	3.6629	169.53	−1.0210	90.78	(35)
4	3.4796	0	−.4321	91.56	3.0156	0	−1.0038	90.10	(19)
10	3.0691	180	−.6799	90.93	3.0043	0	−.9951	89.93	(8)

only one hole j with zero stresses at infinity. In this case, eqns (3.10) become

$$P_n^j = \sum_{\substack{k=1\\ j\neq k}}^{M} \left\{ \sum_{m=-\infty}^{\infty} [A_{nm}^{jk} P_m^k + B_{nm}^{jk} Q_m^k] + A_{n0}^{jk} P_0^\infty + A_{n\,-2}^{jk} P_{-2}^\infty + B_{n\,-2}^{jk} Q_{-2}^\infty \right\},$$

$$Q_n^j = \sum_{\substack{k=1\\ j\neq k}}^{M} \left\{ \sum_{m=-\infty}^{\infty} [C_{nm}^{jk} P_m^k + D_{nm}^{jk} Q_m^k] + C_{n0}^{jk} P_0^\infty + C_{n\,-2}^{jk} P_{-2}^\infty + D_{n\,-2}^{jk} Q_{-2}^\infty \right\}. \quad (3.14)$$

Neglecting high order Fourier coefficients of the pseudotractions, eqns (3.14) are easily solved for the Fourier coefficients of the pseudotractions. The hoop stresses along the surfaces of holes are given by eqn (3.12). Typical results for three equal holes in a row are shown in Table 2. Results coincide with those by Green[7].

Next, we consider an infinite row of equally spaced circular holes of equal radius a. In this case, the pseudotractions are the same for all holes, i.e., $P_n^j = P_n$ and $Q_n^j = Q_n$ for all j, and $\phi^{jk} = 0$ for $j > k$ and $\phi^{jk} = \pi$ for $j < k$. Then, eqns (3.14) become

$$P_{2n} = \sum_{m=-\infty}^{\infty} \left[2 \sum_{k=1}^{\infty} A_{2n2m}^{0k}\right] P_{2m} + \left[2 \sum_{k=1}^{\infty} A_{2n0}^{0k}\right] P_0^\infty + \left[2 \sum_{k=1}^{\infty} A_{2n\,-2}^{0k}\right] P_{-2}^\infty,$$

$$Q_{2n} = \sum_{m=-\infty}^{\infty} \left[2 \sum_{k=1}^{\infty} D_{2n2m}^{0k}\right] Q_{2m} + \left[2 \sum_{k=1}^{\infty} D_{2n\,-2}^{0k}\right] Q_{-2}^\infty, \quad (3.15)$$

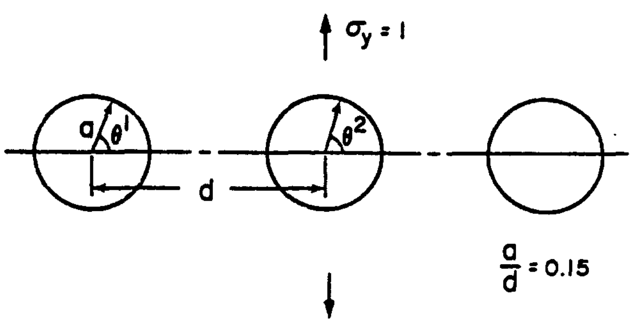
where A_{2k2l}^{0j} and D_{2k2l}^{0j} are given by (3.11) with $d^{0k} = kd$ and

$$I_{2n2m}^{0k} = -\frac{(2n+2m-1)!}{(2n)!(2n-1)!} (a/d)^{2(n+m)} (1/k)^{2(n+m)}. \quad (3.16)$$

Notice that the summation over k produces terms like $\sum (1/k)^{2p}$ which can be easily summed. Then, neglecting high order Fourier coefficients P_n, Q_n, $n < -2N$, and $n >$

Table 2. The hoop stresses in an infinite plate with three equal circular holes in a row.

θ (deg)	σ_θ^1	σ_θ^2	θ (deg)	σ_θ^1
0	2.9960	3.0090	100	−.8074
10	2.8836	2.8962	110	−.4734
20	2.5581	2.5697	120	.0470
30	2.0541	2.0650	130	.6902
40	1.4277	1.4389	140	1.3775
50	.7517	.7652	150	2.0249
60	.1075	.1260	160	2.5535
70	−.4258	−.3990	170	2.8988
80	−.7813	−.7434	180	3.0188
90	−.9143	−.8633		



$2N - 2$, eqns (3.14) become $2N$ equations for $P_{-2N}, P_{-2(N-1)}, \ldots, P_0, \ldots, P_{2(N-1)}$ and $2N - 1$ equations for $Q_{-2N}, \ldots, Q_{-2}, Q_2, \ldots, Q_{2(N-1)}$. Taking $N = 1$ and 2, we obtain the first and second order approximate solutions. For $N = 1$, we have

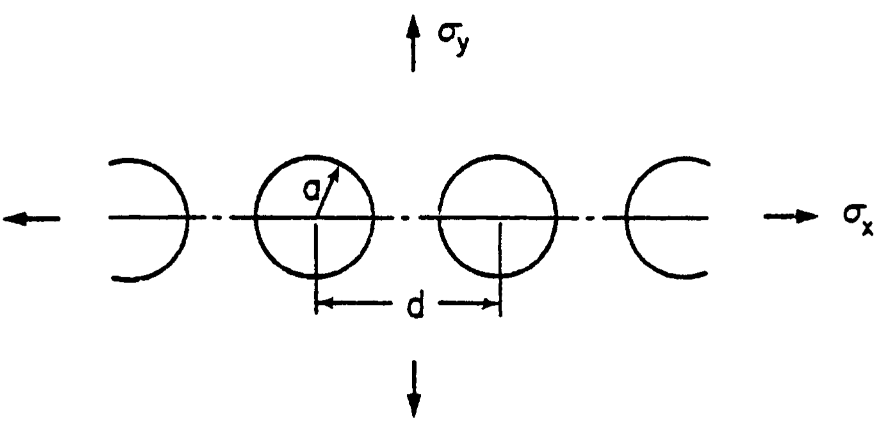
$$\begin{aligned}\sigma_\theta = \{&[(1 + \tfrac{2}{3}\pi^2(a/d)^2 - \tfrac{16}{45}\pi^4(a/d)^4)(\sigma_x^\infty + \sigma_y^\infty) - \tfrac{2}{3}\pi^2(a/d)^2(\sigma_x^\infty - \sigma_y^\infty)] \\ &- 2[-\tfrac{1}{3}\pi^2(a/d)^2(\sigma_x^\infty + \sigma_y^\infty) + \sigma_x^\infty - \sigma_y^\infty]\cos 2\theta\}/[1 + \tfrac{2}{3}\pi^2(a/d)^2 \\ &- \tfrac{16}{45}\pi^4(a/d)^4] - 4\tau_{xy}^\infty \sin 2\theta/[1 - \tfrac{2}{3}\pi^2(a/d)^2 + \tfrac{16}{45}\pi^4(a/d)^4].\end{aligned} \quad (3.17)$$

Results are shown in Table 3. With increasing N, the solutions converge to those by Howland[8]. The approximate solutions show relatively good accuracy.

Table 3. The maximum and minimum stresses in an infinite plate with an infinite row of equal circular holes.

(i) $\sigma_x = 1, \sigma_y = \tau_{xy} = 0$

	σ_θ			Approximate			
				$N = 1$		$N = 2$	
$2a/d$	max	min	(N)	max	min	max	min
.1	2.9363	−.9676	(2)	2.9358	−.9679	2.9363	−.9676
.2	2.7676	−.8759	(3)	2.7608	−.8806	2.7678	−.8760
.3	2.5466	−.7382	(4)	2.517	−.759	2.5480	−.7393
.4	2.3261	−.5699	(5)	2.249	−.627	2.332	−.575
.5	2.1392	−.3866	(5)	1.99	−.50	2.157	−.401
.6	1.9952	−.2085	(5)	1.76	−.39	2.031	−.237
.7	1.8867	−.0681	(8)	1.55	−.30	1.943	−.101
.8	1.8018	−.0029	(8)	1.38	−.24	1.868	−.007



(ii) $\sigma_y = 1, \sigma_x = \tau_{xy} = 0$

	σ_θ			Approximate			
				$N = 1$		$N = 2$	
$2a/d$	max	min	(N)	max	min	max	min
.1	3.0004	−.9682	(2)	3.0006	−.9679	3.0004	−.9682
.2	3.0063	−.8853	(3)	3.0086	−.8806	3.0063	−.8854
.3	3.0308	−.7801	(3)	3.041	−.759	3.0315	−.7812
.4	3.0961	−.6828	(5)	3.120	−.627	3.0995	−.6879
.5	3.2411	−.6124	(5)	3.278	−.501	3.250	−.628
.6	3.5463	−.5739	(6)	3.561	−.391	3.558	−.608
.7	4.2038	−.5605	(8)	4.06	−.30	4.182	−.619
.8	5.7553	−.5617	(9)	5.00	−.24	5.526	−.634

4. AN INFINITELY EXTENDED SOLID WITH CRACKS

In this section, we apply the method of pseudotractions to the problem of an infinitely extended solid containing cracks. The procedure is the same as in the preceding section. First, we consider an infinite solid with two cracks, and then extend our results to many-crack problems.

4.1 *A solid with two cracks*

x^1y^1- and x^2y^2-coordinate systems are employed with origins 0^1 and 0^2 at centers of cracks 1 and 2 of lengths $2c^1$ and $2c^2$, respectively; see Fig. 3. The y^1- and y^2-directions are set to be normal to the crack surfaces C^1 and C^2, respectively. The angle measured from the x^1-direction to the x^2-direction is denoted by θ^{21}. Let the distance between 0^1 and 0^2 be d^{21}, and the angle measured from the x^1-direction to the 0^10^2-direction be denoted by ϕ^{21}. The components in the x^jy^j-coordinates are indicated with the superscript j, $j = 1, 2$. The quantities followed by superscript ∞ denote values at infinity. The original problem is decomposed into a homogeneous problem and sub-problems 1 and 2; see Fig. 4. In the homogeneous problem, an infinite solid without any cracks

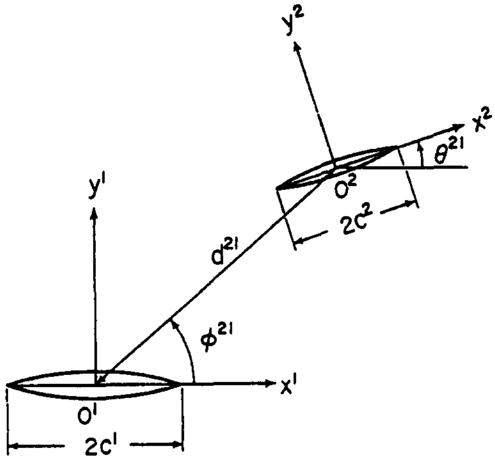


Fig. 3. An infinitely extended plate with two cracks.

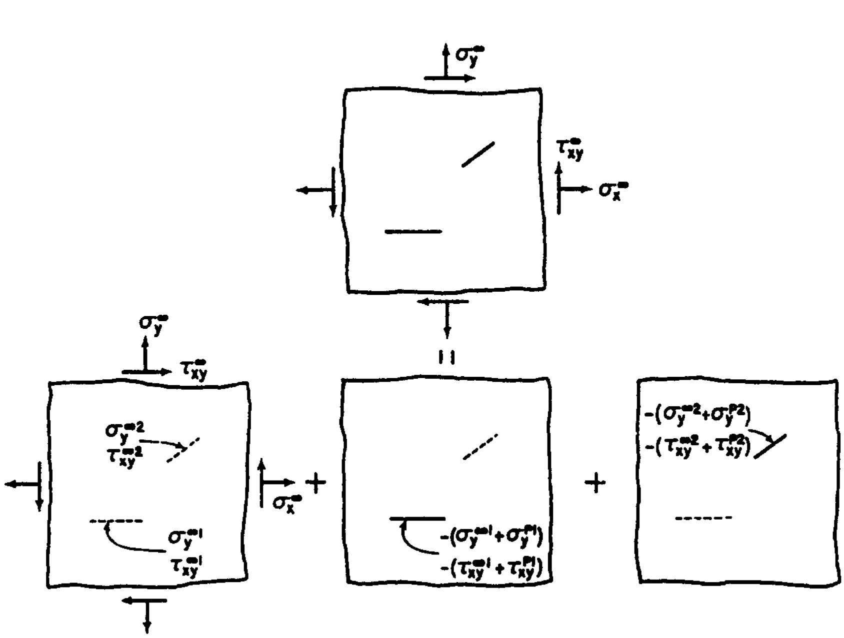


Fig. 4. Decomposition of an original problem into a homogeneous problem and two sub-problems.

is subjected to applied stresses at infinity. In sub-problem j, an infinitely extended solid under zero stresses at infinity has only one crack j, on which the boundary conditions are

$$\sigma_y^j + \sigma_y^{\infty j} + \sigma_y^{Pj} = 0,\ \tau_{xy}^j + \tau_{xy}^{\infty j} + \tau_{xy}^{Pj} = 0, \text{ on } C^j, \qquad j = 1, 2. \tag{4.1}$$

The quantities σ_y^{P1}, τ_{xy}^{P1}, σ_y^{P2}, and τ_{xy}^{P2} are the "pseudotractions." They are the unknown functions which must be determined in such a manner that all boundary conditions of the original problem are satisfied. In sub-problem j, the stress functions are given, as in Muskhelishvili[4], by

$$\begin{aligned}\Phi^{j\prime}(z^j) &= -\frac{1}{2\pi i(z^{j2} - c^{j2})^{1/2}}\int_{-c^j}^{c^j} \frac{(t^2 - c^{j2})^{1/2}}{t - z^j} \\ &\quad \times [\sigma_y^{\infty j} + \sigma_y^{Pj} - i(\tau_{xy}^{\infty j} + \tau_{xy}^{Pj})]\,\mathrm{d}t, \\ \Psi^{j\prime}(z^j) &= \Phi^{j\prime}(\bar{z}^j) - \Phi^{j\prime}(z^j) - z^j\Phi^{j\prime\prime}(z^j), \qquad j = 1, 2,\end{aligned} \tag{4.2}$$

where $z^j = x^j + iy^j$, and primes denote derivatives with respect to the corresponding argument. The requirement that the sum of the homogeneous problem and the sub-problems must be equivalent to the original problem, i.e. the "consistency requirement," leads to

$$\begin{aligned}\sigma_y^{P2} - i\tau_{xy}^{P2} &= \Phi^{1\prime}(z^1) + \overline{\Phi^{1\prime}(z^1)} + e^{-i2\theta^{21}}[z^1\overline{\Phi^{1\prime\prime}(z^1)} + \overline{\Psi^{1\prime}(z^1)}], \\ \sigma_y^{P1} - i\tau_{xy}^{P1} &= \Phi^{2\prime}(z^2) + \overline{\Phi^{2\prime}(z^2)} + e^{-i2\theta^{12}}[z^2\overline{\Phi^{2\prime\prime}(z^2)} + \overline{\Psi^{2\prime}(z^2)}],\end{aligned} \tag{4.3}$$

with

$$\begin{gathered} z^1 = d^{21}e^{i\phi^{21}} + x^2e^{i\theta^{21}},\ z^2 = d^{12}e^{i\theta^{12}} + x^1e^{i\theta^{12}}, \\ d^{12} = d^{21},\ \theta^{12} = -\theta^{21},\ \phi^{12} = \phi^{21} + \pi - \theta^{21}, \\ -c^1 < x^1 < c^1,\ -c^2 < x^2 < c^2. \end{gathered} \tag{4.4}$$

The right-hand sides of eqns (4.3) represent the tractions acting on C^2 and C^1 in sub-problems 1 and 2, respectively. It is seen from eqns (4.1) that eqns (4.3) ensure the stress-free condition on the surfaces of the cracks when the homogeneous problem and the sub-problems are superimposed. Equations (4.3) form a system of integral equations for the pseudotractions, σ_y^{P1}, τ_{xy}^{P1}, and σ_y^{P2}, τ_{xy}^{P2}, which are functions of x^1 and x^2, respectively. In general, it is not easy to solve the system of integral equations (4.3) explicitly. Here again, this system of integral equations can be reduced to a system of algebraic equations in the manner discussed before.

To this end, we expand the pseudotractions into a Taylor series as

$$\sigma_y^{Pj} - i\tau_{xy}^{Pj} = \sum_{n=0}^{\infty} (P_n^j - iQ_n^j)(x^j/c^j)^n, \qquad j = 1, 2. \tag{4.5}$$

Substituting eqns (4.5) into (4.2), we obtain

$$\begin{aligned}\Phi^{j\prime}(z^j) &= \sum_{m=0}^{\infty} (P_{2m}^j - iQ_{2m}^j)\left[\sum_{k=1}^{\infty} f_{mk}(c^j/z^j)^{2k}\right] + \sum_{m=1}^{\infty} (P_{2m-1}^j \\ &\quad - iQ_{2m-1}^j)\left[\sum_{k=1}^{\infty} f_{mk}(c^j/z^j)^{2k+1}\right] + (\sigma_y^{\infty j} - i\tau_{xy}^{\infty j})\left[\sum_{k=1}^{\infty} f_{0k}(c^j/z^j)^{2k}\right],\end{aligned} \tag{4.6}$$

where

$$f_{mk} = g_m \frac{(2k)!}{2^{2k}(m+k)k!(k-1)!}, \quad g_m = \begin{cases} \dfrac{(2m-1)!}{2^{2m}m!(m-1)!} & m > 0 \\ 1/2 & m = 0 \end{cases}. \tag{4.7}$$

Then the consistency condition (4.3) is reduced to a system of algebraic equations,

$$\begin{aligned} P_n^j &= \sum_{m=0}^{\infty} [A_{nm}^{jk} P_m^k + B_{nm}^{jk} Q_m^k] + A_{n0}^{jk}\sigma_y^{\infty k} + B_{n0}^{jk}\tau_{xy}^{\infty k}, \\ Q_n^j &= \sum_{m=0}^{\infty} [C_{nm}^{jk} P_m^k + D_{nm}^{jk} Q_m^k] + C_{n0}^{jk}\sigma_y^{\infty k} + D_{n0}^{jk}\tau_{xy}^{\infty k}, \qquad j, k = 1, 2, j \neq k, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} A_{n2m}^{jk} &= g_m (c^j/d^{jk})^n \sum_{t=1}^{\infty} \frac{h_{nt}}{m+t} (c^k/d^{jk})^{2t} a_{n2t}^{jk}, \\ A_{n\,2m-1}^{jk} &= g_m (c^j/d^{jk})^n \sum_{t=1}^{\infty} \frac{h_{nt}}{m+t} \frac{2t+n}{2t} (c^k/d^{jk})^{2t+1} a_{n\,2t+1}^{jk}, \end{aligned} \tag{4.9}$$

with

$$\begin{aligned} a_{nt}^{jk} &= (n+2)\cos[t\phi^{jk} + n(\phi^{jk} - \theta^{jk})] - (t+n)\cos[t\phi^{jk} + (n+2)(\phi^{jk} - \theta^{jk})] \\ &\quad + t\cos[(t-2)\phi^{jk} + (n+2)(\phi^{jk} - \theta^{jk})], \\ b_{nt}^{jk} &= -(n+2)\sin[t\phi^{jk} + n(\phi^{jk} - \theta^{jk})] + (t+n)\sin[t\phi^{jk} + (n+2)(\phi^{jk} - \theta^{jk})] \\ &\quad - (t-2)\sin[(t-2)\phi^{jk} + (n+2)(\phi^{jk} - \theta^{jk})], \\ c_{nt}^{jk} &= -n\sin[t\phi^{jk} + n(\phi^{jk} - \theta^{jk})] + (t+n)\sin[t\phi^{jk} + (n+2)(\phi^{jk} - \theta^{jk})] \\ &\quad - t\sin[(t-2)\phi^{jk} + (n+2)(\phi^{jk} - \theta^{jk})], \\ d_{nt}^{jk} &= -n\cos[t\phi^{jk} + n(\phi^{jk} - \theta^{jk})] + (t+n)\cos[t\phi^{jk} + (n+2)(\phi^{jk} - \theta^{jk})] \\ &\quad - (t-2)\cos[(t-2)\phi^{jk} + (n+2)(\phi^{jk} - \theta^{jk})], \\ h_{nt} &= (-1)^n \frac{(n+2t-1)!}{2^{2t-1}n![(t-1)!]^2}. \end{aligned} \tag{4.10}$$

The expressions for B_{nt}^{jk}, C_{nt}^{jk}, and D_{nt}^{jk} in eqns (4.8) are obtained from the expression for A_{nt}^{jk} given by (4.9), if we replace in the right-hand side of (4.9), a by b, c, and d, respectively. As is seen from eqns (4.8–4.10), P_n^j and Q_n^j are of the order of $(c/d)^{n+2}$, where $0(c/d) = 0(c^1/d^{21}) = 0(c^2/d^{12})$. Neglecting terms of orders higher than $(c/d)^{N+1}$, we observe that eqns (4.8) are $4N$ linear algebraic equations for $P_0^j, \ldots, P_{N-1}^j$, $Q_0^j, \ldots, Q_{N-1}^j$, $j = 1, 2$. These equations are then easily solved. The stress functions for sub-problems 1 and 2 are given by eqn (4.6), and the solution of the original problem is obtained by superposing those of the homogeneous problem and sub-problems 1 and 2. The stress intensity factors at the tips of the cracks are given by

$$\begin{aligned} K_{I,t=\pm c^j}^j &= \sqrt{\pi c^j}\left[\sigma_y^{\infty j} + P_0^j + \sum_{k=1}^{\infty} (P_{2k}^j \pm P_{2k-1}^j)\frac{(2k)!}{2^{2k}(k!)^2}\right], \\ K_{II,t=\pm c^j}^j &= \sqrt{\pi c^j}\left[\tau_{xy}^{\infty j} + Q_0^j + \sum_{k=1}^{\infty} (Q_{2k}^j \pm Q_{2k-1}^j)\frac{(2k)!}{2^{2k}(k!)^2}\right]. \end{aligned} \tag{4.11}$$

As a typical example, we consider two collinear cracks with equal length, $2c$, separated by distance d. In this case, $c^1 = c^2 = c$, $d^{21} = d$, $\theta^{21} = \theta^{12} = 0$, $\phi^{21} = 0$, $\phi^{12} = \pi$, $P^1_{2t} = P^2_{2t} = P_{2t}$, $P^1_{2t-1} = -P^2_{2t-1} = P_{2t-1}$, $Q^1_{2t} = Q^2_{2t} = Q_{2t}$, and $Q^1_{2t-1} = -Q^2_{2t-1} = Q_{2t-1}$. Hence eqns (4.8) become

$$(-1)^n P_n = \sum_{m=0}^{\infty} A^{21}_{nm} P_m + A^{21}_{n0}\sigma_y^{\infty},$$
$$(-1)^n Q_n = \sum_{m=0}^{\infty} A^{21}_{nm} Q_m + A^{21}_{n0}\tau_{xy}^{\infty}, \qquad (4.12)$$

where A^{21}_{nm} are given by eqns (4.9) with $a^{21}_{nm} = 2$. The stress intensity factors at the two tips of the crack are given in Table 4. The results converge to the exact solution of Erdogan[9], as the number of terms, N, is increased. Within the number of significant figures shown in Table 4, the accuracy of the solution does not change for N greater than for those indicated.

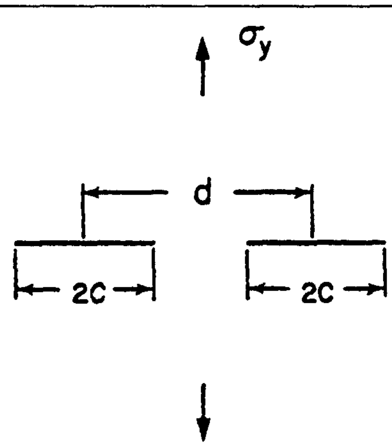
4.2 *A solid with several cracks*

The problem of an infinitely extended solid containing M cracks is solved in a similar manner. The original problem is decomposed into a homogeneous problem and M sub-problems where the typical sub-problem j consists of an infinitely extended solid with only one crack j with zero stresses at infinity. In this case, eqns (4.8) become

$$P^j_n = \sum_{\substack{k=1\\ j\neq k}}^{M} \left\{ \sum_{m=0}^{\infty} [A^{jk}_{nm} P^k_m + B^{jk}_{nm} Q^k_m] + A^{jk}_{n0}\sigma_y^{\infty k} + B^{jk}_{n0}\tau_{xy}^{\infty k} \right\},$$
$$Q^j_n = \sum_{\substack{k=1\\ j\neq k}}^{M} \left\{ \sum_{m=0}^{\infty} [C^{jk}_{nm} P^k_m + D^{jk}_{nm} Q^k_m] + C^{jk}_{n0}\sigma_y^{\infty k} + D^{jk}_{n0}\tau_{xy}^{\infty k} \right\}, \quad j = 1, \ldots, M. \qquad (4.13)$$

Table 4. The stress intensity factor for two equal collinear cracks.

$2c/d$	$K_I/\sigma_y\sqrt{\pi c}$ inside	outside	(N)
0.1	1.0013	1.0012	(2)
0.2	1.0057	1.0046	(4)
0.3	1.0138	1.0102	(5)
0.4	1.0272	1.0179	(7)
0.5	1.0480	1.0280	(9)
0.6	1.0804	1.0409	(12)
0.7	1.1333	1.0579	(21)
0.8	1.2289	1.0811	(28)



For an infinitely extended solid containing an infinite row of collinear and parallel cracks of equal size $2c$ and equal spacing d, the pseudotractions are the same for all cracks. Then we have $P^j_n = P_n$ and $Q^j_n = Q_n$ for all j. Note that $c^j = c$, $\theta^{jk} = 0$, and for collinear cracks, $\phi^{jk} = 0$ for $j > k$ and $\phi^{jk} = \pi$ for $j < k$; and for parallel cracks, $\phi^{jk} = \pi/2$ for $j > k$ and $\phi^{jk} = -\pi/2$ for $j < k$. Then eqns (4.13) become

$$P_{2n} = \sum_{m=0}^{\infty}\left[2\sum_{k=1}^{\infty} A^{0k}_{2n2m}\right] P_{2m} + \left[2\sum_{k=1}^{\infty} A^{0k}_{2n0}\right]\sigma^{\infty}_y ,$$
$$Q_{2n} = \sum_{m=0}^{\infty}\left[2\sum_{k=1}^{\infty} D^{0k}_{2n2m}\right] Q_{2m} + \left[2\sum_{k=1}^{\infty} D^{0k}_{2n0}\right]\tau^{\infty}_{xy}, \quad P_{2n+1} = Q_{2n+1} = 0, \tag{4.14}$$

where A's and D's are given by (4.26) with $d^{0k} = kd$ and

$$a^{0k}_{2n2m} = d^{0k}_{2n2m} = 2, \qquad \text{for collinear cracks, and}$$
$$a^{0k}_{2n2m} = 2(2n + 2m + 1)(-1)^{n+m}, \tag{4.15}$$
$$d^{0k}_{2n2m} = 2(1 - 2n - 2m)(-1)^{n+m}, \qquad \text{for parallel cracks.}$$

Note that summation over k in eqns (4.14) produces terms like $\sum (1/k)^{2p}$. Neglecting terms of orders higher than $(c/d)^{2N}$, eqns (4.14) become two sets of N equations for $P_0, P_2, \ldots, P_{2(N-1)}$, and $Q_0, Q_2, \ldots, Q_{2(N-1)}$. Solving eqns (4.14), we obtain the stress intensity factors at the tips of the cracks by eqns (4.11). Taking $N = 1$ and 2, we have the following first-order and second-order approximate solutions, for collinear cracks:

$$\begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} = \sqrt{\pi c}\Big/\left[1 - \frac{\pi^2}{6}\left(\frac{c}{d}\right)^2\right]\begin{Bmatrix} \sigma^{\infty}_y \\ \tau^{\infty}_{xy} \end{Bmatrix}, \qquad (N = 1), \tag{4.16}$$
$$\begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} = \sqrt{\pi c}\left[1 + \frac{\pi^4}{60}\left(\frac{c}{d}\right)^4\right]\Big/\left[1 - \frac{\pi^2}{6}\left(\frac{c}{d}\right)^2 - \frac{\pi^4}{120}\left(\frac{c}{d}\right)^4\right]\begin{Bmatrix} \sigma^{\infty}_y \\ \tau^{\infty}_{xy} \end{Bmatrix}, \qquad (N = 2),$$

and for parallel cracks:

$$K_I = \sqrt{\pi c}\,\sigma^{\infty}_y\Big/\left[1 + \frac{\pi^2}{2}\left(\frac{c}{d}\right)^2\right],$$
$$K_{II} = \sqrt{\pi c}\,\tau^{\infty}_{xy}\Big/\left[1 - \frac{\pi^2}{6}\left(\frac{c}{d}\right)^2\right], \qquad (N = 1),$$
$$K_I = \sqrt{\pi c}\,\sigma^{\infty}_y\left[1 + \frac{\pi^4}{16}\left(\frac{c}{d}\right)^4\right]\Big/\left[1 + \frac{\pi^2}{2}\left(\frac{c}{d}\right)^2 - \frac{\pi^4}{16}\left(\frac{c}{d}\right)^4\right], \tag{4.17}$$
$$K_{II} = \sqrt{\pi c}\,\tau^{\infty}_{xy}\left[1 - \frac{\pi^4}{40}\left(\frac{c}{d}\right)^4\right]\Big/\left[1 - \frac{\pi^2}{6}\left(\frac{c}{d}\right)^2 + \frac{\pi^4}{20}\left(\frac{c}{d}\right)^4\right], \qquad (N = 2).$$

Typical results are shown in Table 5. The solution converges with increasing number of terms, N. For an infinite row of collinear cracks, the solution converges to the exact solution of Westergaard[10]. For an infinite row of parallel cracks, results compare well with Fig. 2.29 (p. 123) of Isida[11]. Simple approximate solutions show relatively good accuracy.

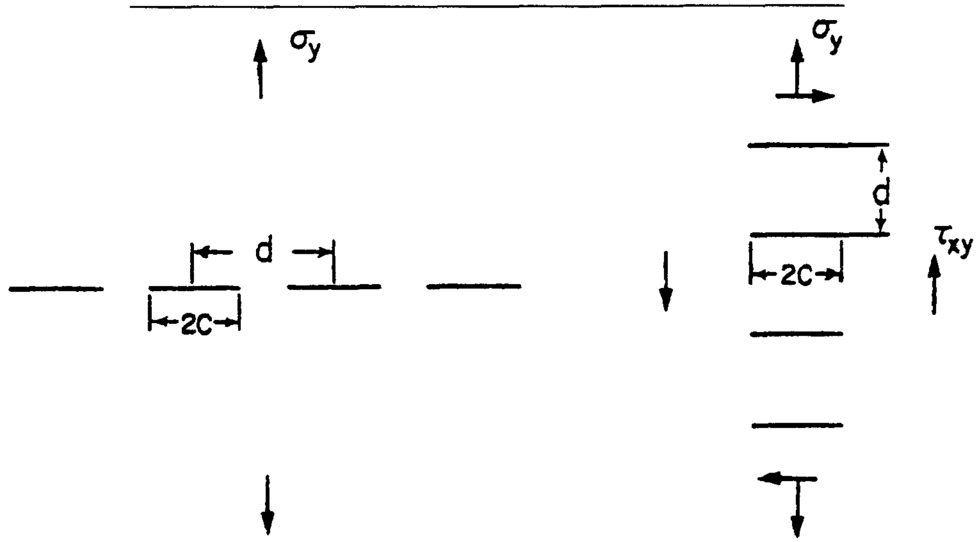
5. DISCUSSION

The method of pseudotractions illustrated above is applicable to various kinds of inhomogeneities other than cracks and holes. By this method, problems of an infinitely

Table 5. The stress intensity factor for an infinite row of collinear and parallel cracks.

(i) An infinite row of collinear cracks.

$2c/d$	$K_I/\sigma_y\sqrt{\pi c}$	(N)	Approximate $N = 1$	Approximate $N = 2$
0.1	1.0041	(1)	1.0041	1.0041
0.2	1.0170	(2)	1.0167	1.0170
0.3	1.0398	(3)	1.0384	1.0397
0.4	1.0753	(3)	1.0704	1.0747
0.5	1.1284	(5)	1.115	1.1256
0.6	1.2085	(8)	1.17	1.1985
0.7	1.3360	(10)	1.25	1.303
0.8	1.5650	(17)	1.36	1.45



(ii) An infinite row of parallel cracks.

$2c/d$	$K_I/\sigma_y\sqrt{\pi c}$	(N)	Approximate $N = 1$	Approximate $N = 2$
0.1	.9879	(2)	.9878	.9879
0.2	.9541	(2)	.9530	.9541
0.3	.9049	(4)	.9001	.9053
0.4	.8479	(4)	.835	.8502
0.5	.7896	(6)	.764	.797
0.6	.7344	(8)	.692	.752
0.7	.6845	(11)	.623	.721
0.8	.6407	(19)	.56	.708

$2c/d$	$K_{II}/\tau_{xy}\sqrt{\pi c}$	(N)	Approximate $N = 1$	Approximate $N = 2$
0.1	1.0041	(1)	1.0041	1.0041
0.2	1.0160	(2)	1.0167	1.0160
0.3	1.0349	(3)	1.0384	1.0345
0.4	1.0593	(4)	1.070	1.0574
0.5	1.0881	(7)	1.115	1.0811
0.6	1.1197	(8)	1.17	1.0997
0.7	1.1532	(12)	1.25	1.105
0.8	1.1877	(19)	1.36	1.09

extended solid containing inhomogeneities are reduced to systems of integral equations for the pseudotractions. In this process, only the solution of an infinitely extended solid with one inhomogeneity is required. Discretizing the pseudotractions, the corresponding system of integral equations is reduced to a system of algebraic equations. These equations involve powers of the ratio of the size of inhomogeneities to their spacing. Neglecting high order terms (when the spacing exceeds the size), we solve the system of algebraic equations. As illustrated for the examples of cracks and holes, the solution converges quickly when the inhomogeneities are suitably apart. For the case of identical inhomogeneities equally spaced in a row, the approximate solution shows relatively good accuracy with only a few leading terms considered.

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